The work of Andrei Okounkov

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Partitions

A partition of n is a non-increasing sequence of integers adding up to n. A partition can be represented by its Young diagram.

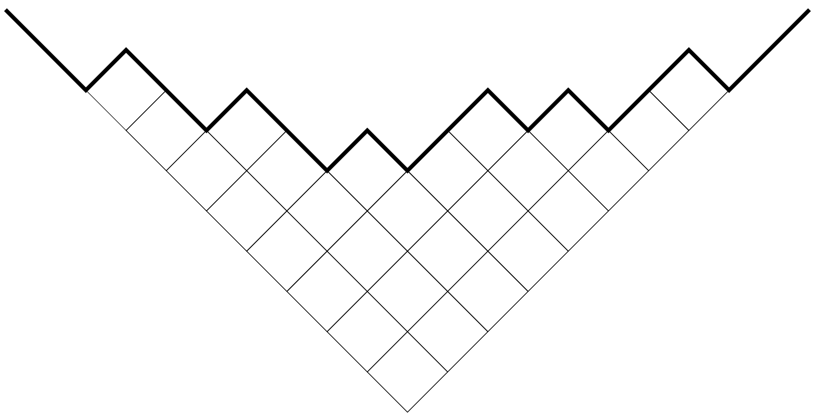
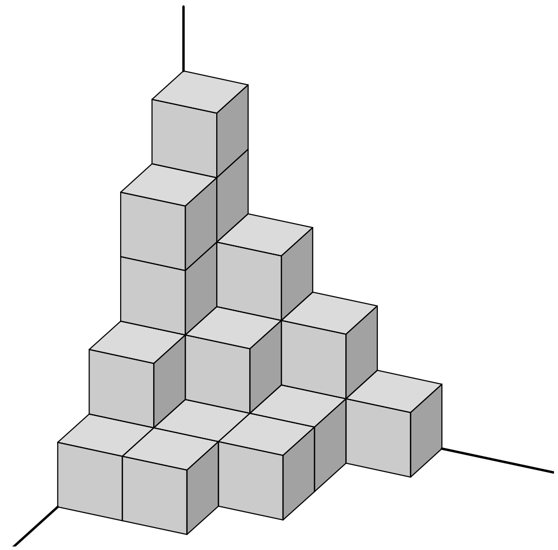


Fig.2 Диаграмма Юнга



A (2D) partition (drawn in the Russian way) and a 3D partition

Partitions are a basic combinatorial object and are at the heart of the theory of the symmetric group $S(n)$.

Gromov–Witten invariants

V a smooth projective variety over $\mathbb{C}$

$\overline{M}_{g,n}(V,\beta) =$ Moduli space of stable maps $(C, p_1, \ldots, p_n, f)$ consisting of a curve C of genus g with marked points p_j and a map $f\colon C \to V$. $[f(C)] = \beta \in H_2(V)$.

Natural cohomology classes in $\overline{M}_{g,n}(V,\beta)$:

1) pull-backs $\mathrm{ev}_i^*\gamma$ of classes $\gamma \in H^*(V,\mathbb{C})$ by $\mathrm{ev}_i\colon (C, p_1, \ldots, p_n, f) \mapsto f(p_i)$.

2)$\psi_i = c_1(L_i) \in H^2(\overline{M}_{g,n}(V,\beta), \mathbb{C})$, fiber of $L_i =$ cotangent space to C at p_i.

Gromov–Witten invariants are intersection numbers

$$\langle \tau_{k_1}(\gamma_1)\cdots\tau_{k_n}(\gamma_n)\rangle^V_{\beta,g} = \int_{\overline{M}_{g,n}(V,\beta)} \prod_{j=1}^{n} \psi_i^{k_i} \mathrm{ev}_i^* \gamma_i \quad (\gamma_i \in H^*(V))$$

If $k_i = 0$ they have the interpretation of numbers of curves of fixed genus and degree passing through given submanifolds.

The theory is already deep and non-trivial for V a point. Witten conjectured and Kontsevich proved that the generating function

$$F(t_0, t_1, \dots) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k_1+\cdots+k_n=3g-3+n} \langle \tau_{k_1}\cdots\tau_{k_n}\rangle_g^{\mathrm{pt}} \prod_{j=1}^{n} t_{k_j},$$

obeys an infinite set of pde's (Virasoro conditions)

$$L_k(e^F) = 0, \qquad k = -1, 0, 1, 2 \dots, \qquad [L_j, L_k] = (j-k)L_{j+k}$$

for certain differential operators L_k of order at most 2.

Gromov–Witten/Hurwitz correspondence

Okounkov and Pandharipande gave an exhaustive description of the GW invariants of curves V. Set $\beta = d \cdot [V]$.

Basic formula for $\omega =$ area form, as a sum over partition of d.

$$\langle \tau_{k_1}(\omega) \cdots \tau_{k_n}(\omega) \rangle_d^{\bullet V} = \sum_{|\lambda|=d} \left(\frac{\dim \lambda}{d!} \right)^{2-2g(V)} \prod_{i=1}^{n} \frac{p_{k_i+1}(\lambda)}{(k_i+1)!}.$$

Cf. Burnside formula for Hurwitz numbers of degree d coverings of V with ramifications $\sim z^{k_i+1}$ at prescribed points:

$$H_d^V(k_1+1, \ldots, k_n+1) = \sum_{|\lambda|=d} \left(\frac{\dim \lambda}{d!} \right)^{2-2g(V)} \prod_{i=1}^{n} \frac{f_{k_i+1}(\lambda)}{(k_i+1)!}.$$

The functions f_k, p_k are sets of generators of the algebra of shifted symmetric functions on the set of all partitions.

Shifted symmetric functions (Kerov, Olshanski, Okounkov)

$P(n)$ = set of partitions of n, $P = \cup P(n)$. $Z(n)$ = center of $\mathbb{Q}S(n)$.

The algebra of shifted symmetric functions Λ^* is the image of the Fourier transform ϕ: $\lim_{\to} Z(n) \to \mathbb{Q}^P$. It consists of polynomials $\mathbb{Q}[\lambda_1, \lambda_2, \cdots]$ invariant under permutations of $\lambda_i - i$.

Hurwitz: $f_n = \phi(C_{(n)})$, the image of the conjugacy class $(1...n)$.

Gromov–Witten: regularized power sums

$$p_n(\lambda) = \sum_{j=1}^{\infty} \left(\left(\lambda_j - j + \tfrac{1}{2} \right)^n - \left(-j + \tfrac{1}{2} \right)^n \right) + (1 - 2^{-n})\zeta(-n)$$

The second and third terms "cancel out" in the spirit of Ramanujan's $1 + 2 + 3 + \cdots = -\frac{1}{12}$.

Elliptic curves and rational billiards

If $g(V) = 0, 1$, Okounkov and Pandharipande express generating functions in terms of infinite wedge representations (fermionic Fock spaces). For $g(V) = 1$, by a result of Bloch-Okounkov, these generating functions are expressed in terms of Jacobi theta functions. It follows that

$$\sum_d q^d \langle \tau_{k_1}(\omega) \cdots \tau_{k_n}(\omega) \rangle_d^{\bullet V} \in \mathbb{Q}(E_2, E_4, E_6)$$

i.e., it is a quasimodular form.

Eskin and Okounkov compute the $d \to \infty$ asymptotic yielding the volumes of the space of holomorphic differentials on a curve whose zeros have given multiplicities. The latter is related to the dynamics of rational billiards (Eskin, Mazur, Kontsevich, Zorich).

Other uses of partitions

Baik–Deift–Johansson conjecture: the joint distribution of the first few eigenvalues of a random hermitian matrix is the same (after rescaling) as the joint distribution of first few longest increasing subsequence in a random permutation. (Okounkov 1999)

Instanton sum in $\mathcal{N} = 2$ supersymmetric gauge theory is described by random partitions with a periodic potential. This gives a derivation of the Seiberg–Witten prepotential from gauge theory and explains the connection with integrable systems. (Nekrasov–Okounkov 2003)

Ring structure of the quantum cohomology of the Hilbert scheme of points in $\mathbb{C}^2$ and time-dependent quantum Calogero–Moser systems. (Okounkov–Pandharipande 2004)

Gromov–Witten and Donaldson–Thomas

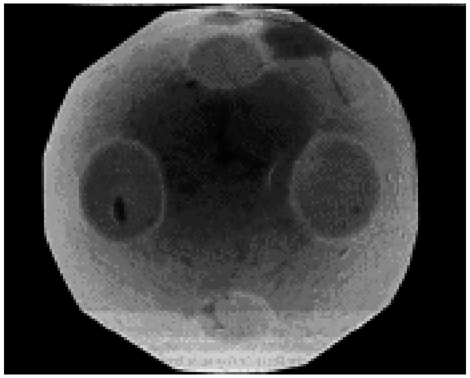
Gromov-Witten invariants are special (and difficult to compute) for V a three-fold, the case originally studied by physicists.

Conjectures of Maulik, Nekrasov, Okounkov and Pandharipande (proven in several cases): 1) generating functions of GW invariant are related to Donaldson–Thomas invariants:

$$(-iu)^{-d}Z'_{GW}(\gamma_1,\ldots,\gamma_n;u)_\beta = q^{-d/2}Z'_{DT}(\gamma_1,\ldots,\gamma_n;q)_\beta, \quad q=-e^{iu},$$

where $d=-\beta\cdot K_V$ is the virtual dimension. 2) $Z'_{DT}(\gamma;q)_\beta$ is a *rational* function of q.

DT invariants are intersection numbers on the Hilbert scheme of 1-dimensional subschemes of V.

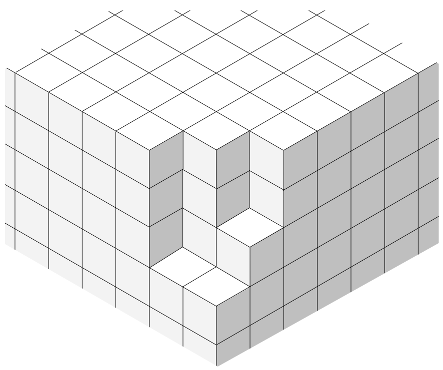


A gold crystal at 1000 °C, Heyraud and Métois, 1980

Dimer models and random surfaces

A dimer configuration (or perfect matching) on a bipartite graph G is a subset of the set of edges of G meeting every vertex exactly once.

A dimer configuration on a planar graph can be thought of as a surface. Example: honeycomb lattice with dimers replaced by rhombi.



Planar dimer models

Kenyon, Okounkov and Sheffield models: dimers on doubly periodic bipartite graphs G in the plane with positive weight assignements to the edges.

Question from statistical mechanics: asymptotic behaviour of the measure on dimer configurations on $G_n = G/n\mathbb{Z}^2$ as $n \to \infty$.

Basic object: the spectral curve

$$\{z, w \in (\mathbb{C}^\times)^2,\ P(z,w) = 0\}$$

where $P(z,w) = \det K(z,w)$ is the determinant of the Kasteleyn matrix (weighted connectivity matrix) of G_1 twisted by a character (z,w) of $\mathbb{Z}^2$.

Planar dimer models and Harnack curves

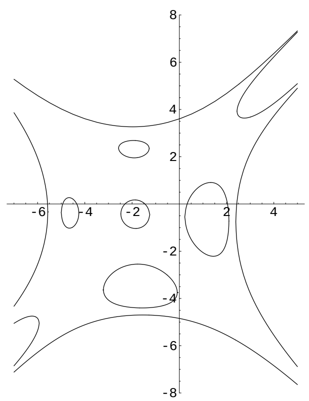
The amoeba of a curve $C \subset \mathbb{C}^2$ is the image of C under the map Log: $(z, w) \to (\log|z|, \log|w|)$.

Kenyon, Okounkov and Sheffield prove that the spectral curve belongs to the class of Harnack curves.

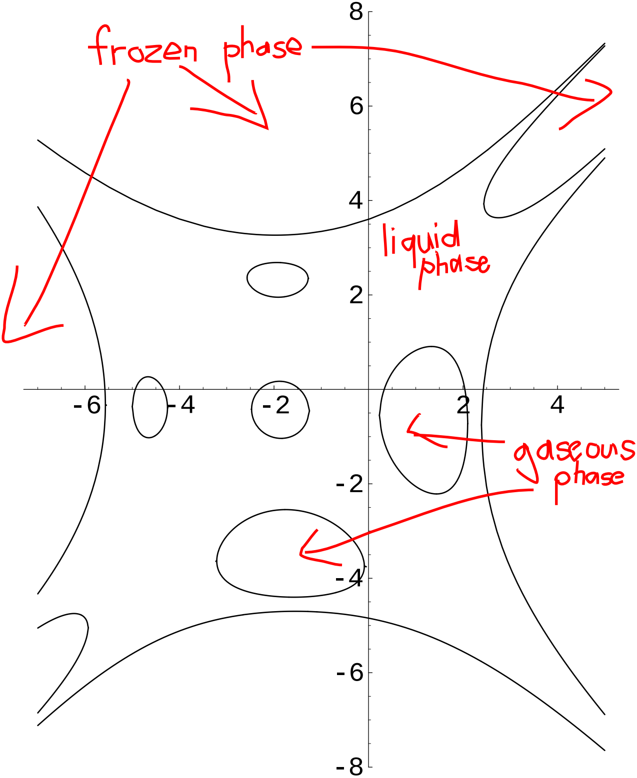
Harnack curves play a distinguished role in modern real algebraic geometry. They can be characterized by the property that Log is $2:1$ on the interior of their amoeba (Mikhalkin). Kenyon and Okounkov show that *all* Harnack curves are obtained from some dimer weights and give thus a parametrization of their moduli space.

Phase diagram

Here is an amoeba ...



Weights of dimer models come in two parameter families depending on the "magnetic field" (x, y) with spectral curve $P(e^x z, e^y w) = 0$. As a function of (x, y) the variance of the height difference $h(a) - h(b)$ as $|a - b| \to \infty$ has three qualitatively different behaviours (phases) depending on where (x, y) is with respect to the amoeba.

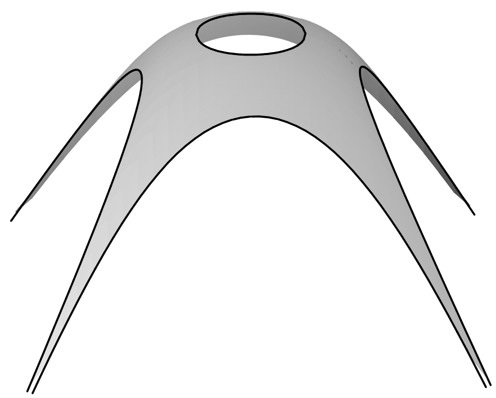

frozen phase
liquid phase
gaseous phase
8
6
4
2
-2
-4
-6
-8
-6
-4
-2
2
4

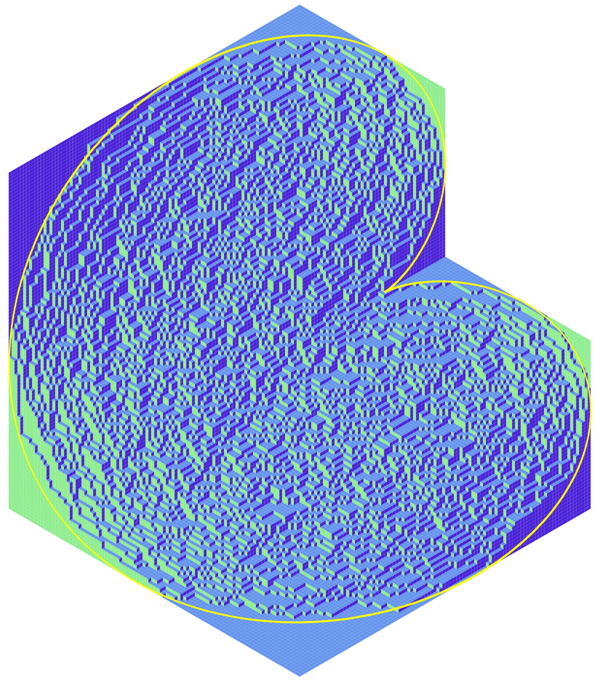
Equilibrium shapes, limiting shapes

The equilibrium shape with "crystal corner" boundary condition is given by the graph of (minus) the Ronkin function

$$h(x,y) = \frac{1}{2\pi^2}\int_{S^1\times S^1} \log P(e^x z, e^y w)\frac{dzdw}{zw},$$

which has facets (flat pieces) on the complement of the amoeba. Genus one example (square-octagon lattice):





The cardioid as the frozen boundary of the dimer model on the honeycomb lattice (Kenyon–Okounkov)