

## The work of Grigory Perelman

## Grigory Perelman

Born 1966

PhD from St. Petersburg State University

Riemannian geometry and Alexandrov geometry

1994 ICM talk

## Soul Conjecture

Conjectured by Cheeger-Gromoll, 1972

Proved by Perelman, 1994

If  $M$  is a complete noncompact Riemannian manifold with nonnegative sectional curvature, and there is one point where all of the sectional curvatures are positive, then  $M$  is diffeomorphic to Euclidean space.

## Poincare Conjecture (1904)

A simply-connected compact three-dimensional manifold is diffeomorphic to the three-sphere.

## Geometrization Conjecture (Thurston, 1970's)

A compact orientable three-dimensional manifold can be canonically cut along two-dimensional spheres and two-dimensional tori into "geometric pieces".

## Ricci flow approach to the Poincare and Geometrization Conjectures

Ricci flow equation introduced by Hamilton (1982)

Program to prove the conjectures using Ricci flow : Hamilton and Yau

## Perelman's Ricci flow papers

(November, 2002)

The entropy formula for the Ricci flow and its geometric applications

(March, 2003)

Ricci flow with surgery on three-manifolds

(July, 2003)

Finite extinction time for the solutions to the Ricci flow on certain three-manifolds

## Detailed expositions of Perelman's work

- Cao-Zhu
- Kleiner-Lott
- Morgan-Tian

## Perelman's Ricci flow papers

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## Hamilton's Ricci flow equation

$$\frac{dg}{dt} = -2 Ric$$

$g(t)$  is a 1-parameter family of Riemannian metrics on a manifold  $M$ .

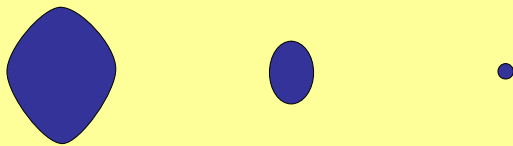
$Ric$  = the Ricci tensor of  $g(t)$

(Assume that  $M$  is three-dimensional, compact and orientable.)

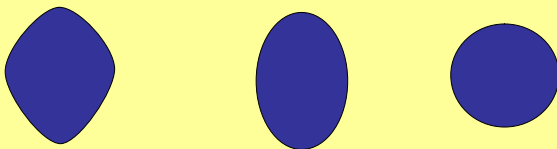
## Theorem (Hamilton 1982)

If a simply-connected compact three-dimensional manifold has a Riemannian metric with positive Ricci curvature then it is diffeomorphic to the 3-sphere.

### Unnormalized Ricci flow



### Normalized Ricci flow



## Hamilton's 3-D nonsingular flows theorem

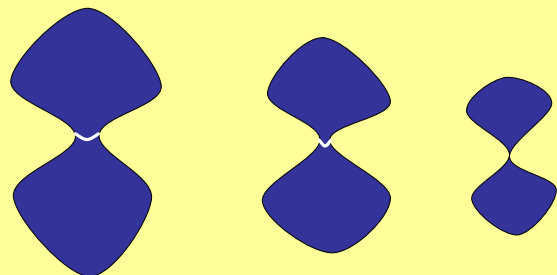
### Theorem (Hamilton 1999) :

Suppose that the normalized Ricci flow on a compact orientable 3-manifold  $M$  has a smooth solution that exists for all positive time and has uniformly bounded sectional curvature. Then  $M$  satisfies the geometrization conjecture.

### Remaining issues :

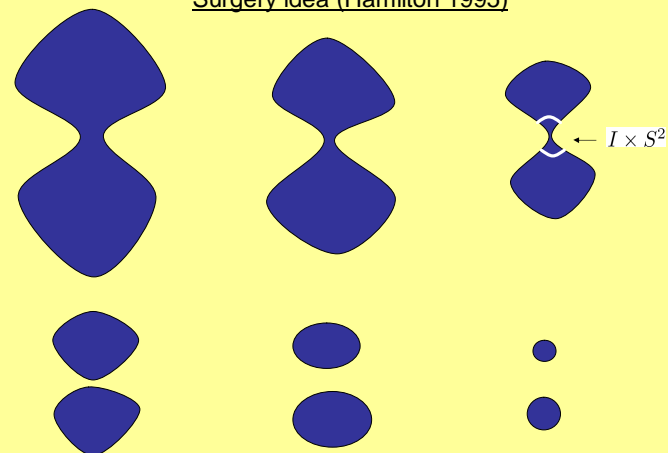
1. How to deal with singularities
2. How to remove the curvature assumption

## Neckpinch singularity



A two-sphere pinches

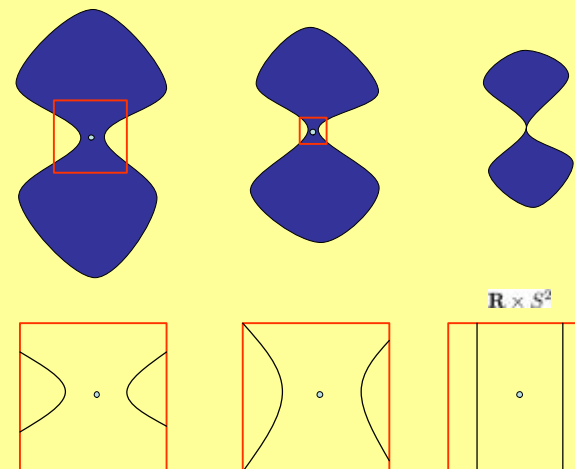
## Surgery idea (Hamilton 1995)



## What are the possible singularities?

Fact : Singularities come from a sectional curvature blowup.

Rescaling method to analyze singularities (Hamilton)



## Blowup analysis

Idea : take a convergent subsequence of the rescaled solutions, to get a limiting Ricci flow solution. This will model the singularity formation.

Does such a limit exist?

If so, it will be very special :

1. It lives for all negative time (ancient solution)
2. It has nonnegative curvature (Hamilton-Ivey)

Hamilton's compactness theorem gives sufficient conditions to extract a convergent subsequence.

In the rescaled solutions, one needs :

1. Uniform curvature bounds on balls.
2. A uniform lower bound on the injectivity radius at the basepoint.

By carefully selecting the blowup points, one gets the curvature bounds.

Two obstacles :

1. How to get the injectivity radius bound?
2. What are the possible blowup limits?

## Three themes of Perelman's work

- No local collapsing theorem
- Ricci flow with surgery
- Long time behavior

## No local collapsing theorem (Perelman1)

Curvature bounds imply injectivity radius bounds.  
(Gives blowup limits.)

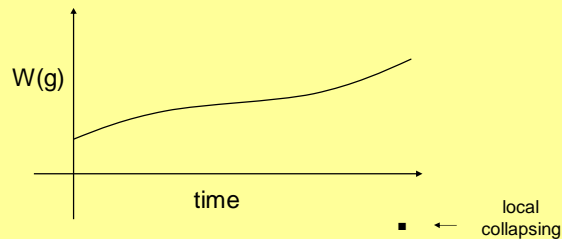
Let  $M$  be a compact  $n$ -dimensional manifold.

If  $g(\cdot)$  is a Ricci flow solution on  $M$  that exists for  $t \in [0, T)$ , with  $T < \infty$ , then for every  $\rho > 0$  there is a  $\kappa > 0$  with the following property.

Suppose that  $r \in (0, \rho)$  and let  $B_t(x, r)$  be a metric  $r$ -ball in a time- $t$  slice. If the sectional curvatures on  $B_t(x, r)$  are bounded in absolute value by  $\frac{1}{r^2}$  then the volume of  $B_t(x, r)$  is bounded below by  $\kappa r^n$ .

## Method of proof

New monotonic quantities for Ricci flow :  
W-entropy, reduced volume



## Classification of 3D blowup limits (Perelman1, Perelman2)

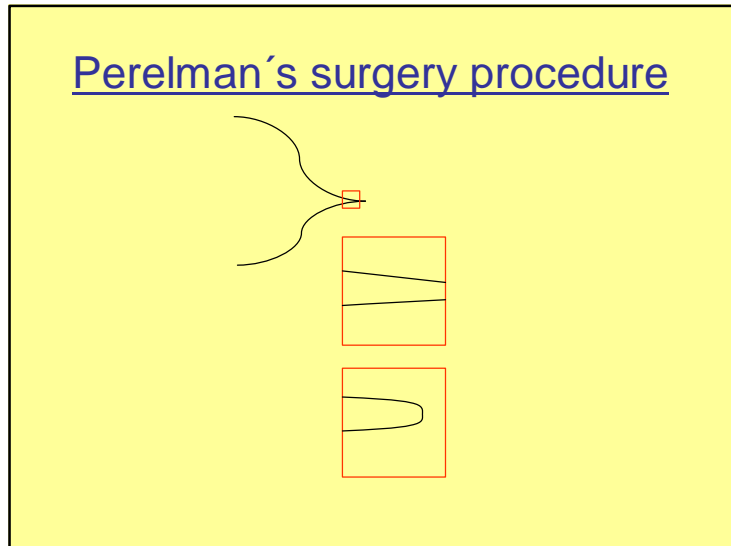
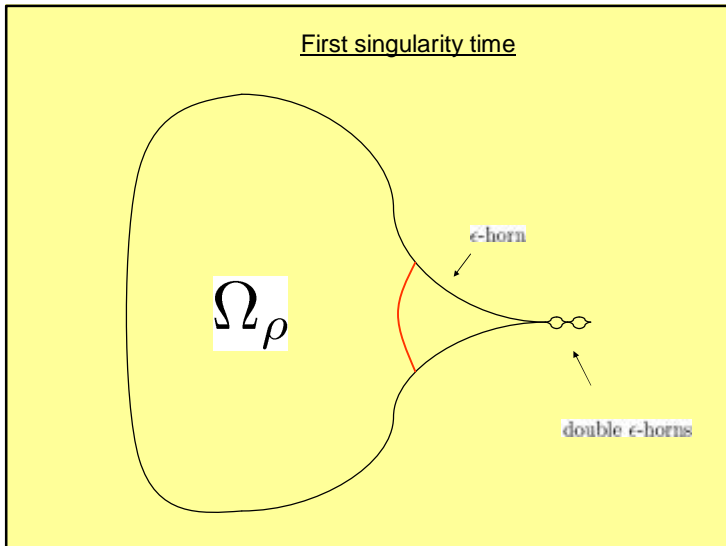
- Finite quotient of the round shrinking 3-sphere
- Diffeomorphic to 3-sphere or real projective space
- Round shrinking cylinder or its  $(\mathbb{Z}/2\mathbb{Z})$ -quotient
- Diffeomorphic to Euclidean 3-space and, after rescaling, each time slice is necklike at infinity

## Canonical neighborhood theorem (Perelman 1)

Any region of high scalar curvature in a 3D Ricci flow is modeled, after rescaling, by the corresponding region in a blowup limit.

## Ricci flow with surgery for three-manifolds

- Find 2-spheres to cut along
- Show that the surgery times do not accumulate



Main problem

At later singularity times, one still needs to find 2-spheres along which to cut.

Still need : ``canonical neighborhood theorem`` and ``no local collapsing theorem``.

But earlier surgeries could invalidate these.

One ingredient of the solution

Perform surgery deep in the epsilon-necks.

End up doing surgery on long thin tubes.

## Surgery theorem (Perelman2)

One can choose the surgery parameters so that there is a well defined Ricci-flow-with-surgery, that exists for all time.

In particular, there is only a finite number of surgeries on each finite time interval.

(Note : There could be an infinite number of total surgeries.)

## Long time behavior

Special case : M simply-connected

Finite extinction time theorem  
(Perelman3, Colding-Minicozzi)

If M is simply-connected then after a finite time, the remaining manifold is the empty set.

Consequence : M is a connected sum of standard pieces (quotients of the round three-sphere and circle x 2-sphere factors). From the simple-connectivity, it is diffeomorphic to a three-sphere.

## Long time behavior

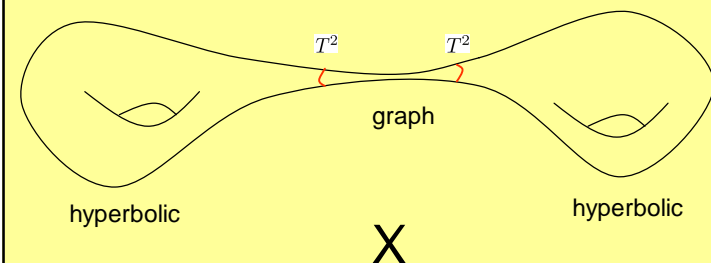
General case : M may not be simply-connected

To see the limiting behavior, rescale

the metric to  $\hat{g}(t) = \frac{g(t)}{t}$ .

X a connected component of the time-t manifold.

## Desired picture



### Perelman's thick-thin decomposition

Thick part of  $X$  :

- Locally volume-noncollapsed
- Local two-sided sectional curvature bound

Thin part of  $X$  :

- Locally volume-collapsed
- Local lower sectional curvature bound

### The thick part becomes hyperbolic

Theorem (Perelman2) :

For large time, the thick part of  $X$  approaches the thick part of a finite-volume manifold of constant sectional curvature  $-1/4$ .

Furthermore, the cuspidal 2-tori (if any) are incompressible in  $X$ .

Based partly on arguments from Hamilton (1999).

### The thin part

Theorem

(Perelman2, Shioya-Yamaguchi)

For large time, the thin part of  $X$  is a graph manifold.

### Upshot

The original manifold  $M$  is a connected sum of pieces  $X$ , each with a hyperbolic/graph decomposition.

Grigory Perelman

Fields Medal 2006

For his contributions to geometry  
and his revolutionary insights into  
the analytical and geometric  
structure of Ricci flow