August 16, 2006

SLIDES

The Work of Terence Tao

Math at its Highest Level Can Make us Think:

- What Amazing Technical Power!
- What a Grand Synthesis!
- How Could Anyone Not Have Seen This Before?
- Where on Earth Did This Idea Come From?

Three Topics Among Many

- Kakeya needle problem
- Non-linear Schrödinger equations
- Arithmetic progressions of primes

Kakeya Needle Problem (Classic Version)

Let $E \subset \mathbb{R}^2$

Suppose we can turn a needle of length 1 by a full 360 degrees, keeping it inside E at all times.

Question: How small can we take the area of E?

Answer: Arbitrarily small - (Besicovitch, Pál)

Kakeya Needle Problem (Modern Version)

Let $E \subset \mathbb{R}^n$. Assume E contains unit line segments in all directions. ("Besicovitch set".)

What can we say about the fractal dimension of E?

“Fractal Dimension” = Minkowski Dimension

Fix $E \subset \mathbb{R}^n, \beta > 0$.

ASK: For small enough $\delta > 0$, can we cover E by $\delta^{-\beta}$ balls of radius δ ?

$$\dim E = \inf\{\text{all } \beta > 0 | \text{ Answer is YES}\}$$

The Kakeya Problem is Related To

- Fourier Analysis
- Partial Differential Equations
- Combinatorics

Theorems on Kakeya:

Any Besicovich set $E \subset \mathbb{R}^n$ has Minkowski dimension at least $\beta(n)$.

Try for the best $\beta(n)$.

Maybe, ultimately, $\beta(n) = n$?

Classic Results of 1970's & 1980's

$\beta(2) = 2$ Davies;

See also A. Córdoba.

$\beta(n) \geq \frac{n+1}{2}$ for $n \geq 3$ Drury;

See also Christ-Duoandikoetxa-Rubio de la Francia

Breakthroughs of the 1990's

- Besicovitch sets of small fractal dimension have geometric structure. (They contain "bouquets" and "hairbrushes".) - J. Bourgain, T. Wolff.

- The Kakeya problem is related to Gowers' improvement of the Balog-Szemerédi Theorem from combinatorics. - J. Bourgain.

- The subject grew deep and forbidding!

Deep Theorem:

(Bourgain, following Gowers' ideas on Balog-Szemerédi)

Let A, B be subsets of an abelian group.

Let $G \subset A \times B$.

Assume $\#A, \#B, \#\{a + b : (a, b) \in G\} \leq N$.

Then $\#\{a - b : (a, b) \in G\} \leq CN^{2-\frac{1}{13}}$,

where C is a universal constant.

Little Lemma (Nets Katz and Terence Tao 1999):

Same Assumptions as the Deep Theorem

imply

$$\{a - b : (a, b) \in G\} \leq CN^{2-\frac{1}{6}},$$

where C is a universal constant.

Corollary: $\beta(n) \geq \frac{4n+3}{7}$,

sharpest result then known for $n > 8$.

Further progress ...

Tour de Force by

Nets Katz, Izabella Laba, Terence Tao

The subject is still deep and forbidding.

Complete solution seems far away.

Interaction Morawetz Estimates

Brought to us by the

I-Team:

J. Colliander,	M. Keel,
G. Staffilani,	H. Takaoka,
and	
T. Tao	

Non-Linear Schrödinger (NLS)

$$\left\{ \begin{array}{l} i\partial_t u + \Delta_x u = \pm |u|^{p-1} u \\ \\ u\big|_{t=0} = u_0 \text{ given.} \end{array} \right\}$$

Here, $u(x,t)$ is a complex-valued function of $x \in \mathbb{R}^3$ and $t \in \mathbb{R}$.

Minus sign $\Longrightarrow$ Focussing

Plus sign $\Longrightarrow$ Defocussing

Obvious Conserved Quantities

- MASS $= \int_{\mathbb{R}^3} |u(x,t)|^2 \, dx$

- ENERGY $= \frac{1}{2} \int_{\mathbb{R}^3} |\nabla_x u(x,t)|^2 \, dx$

$$+ \frac{1}{p+1} \int_{\mathbb{R}^3} |u(x,t)|^{p+1} \, dx$$

Morawetz Estimate (Defocussing NLS, $p = 3$)

$$\int_0^T \int_{\mathbb{R}^3} \frac{|u(x,t)|^4}{|x|} \, dx \, dt \ \leq$$

$$C \sup_{0 \leq t \leq T} \| \ (-\Delta_x)^{1/4} \, u(\cdot, t) \ \|^2_{L^2(\mathbb{R}^3)} \ .$$

- <u>Rules Out:</u> $|u(x,t)| \sim 1$ for $|x| \lesssim 1$, all t.

- <u>Doesn't Rule Out:</u> $|u(x,t)| \sim 1$ for $|x - x_0(t)| \lesssim 1$, all t.

Proof of Morawetz Estimate

- Set $M_0(t) = \text{Im} \int\limits_{\mathbb{R}^3} \bar{u}(x,t) \cdot \left[\frac{x}{|x|} \cdot \nabla_x u(x,t) \right] dx$

- Check $|M_0(t)| \leq C \parallel (-\Delta_x)^{1/4} u(\cdot, t) \parallel^2_{L^2(\mathbb{R}^3)}$

- Compute (using NLS) that $\frac{dM_0(t)}{dt} \quad =$

$$c_1 |u(0,t)|^2 + c_2 \int\limits_{\mathbb{R}^3} |\nabla_\Omega u(x,t)|^2 \frac{dx}{|x|} + \int\limits_{\mathbb{R}^3} \frac{|u(x,t)|^4}{|x|} dx$$

($\nabla_\Omega =$ angular part of the gradient.)

The Problem: Taking

$$M_0(t) = \text{Im} \int_{\mathbb{R}^3} \bar{u}(x,t) \cdot \left[\frac{x}{|x|} \cdot \nabla_x u(x,t) \right] dx$$

picks out the origin. Maybe u lives far away from the origin, so $M_0(t)$ is irrelevant.

The Solution: Move $M_0(t)$ to wherever u lives.

The I-Team defined

$$M(t) = \int_{\mathbb{R}^3} M_y(t) \cdot |u(y,t)|^2 \, dy, \qquad \text{where}$$

$$M_y(t) = \text{Im} \int_{\mathbb{R}^3} \bar{u}(x,t) \cdot \left[\frac{x-y}{|x-y|} \cdot \nabla_x \, u(x,t) \right] dx \, .$$

Rerunning the proof, using $M(t)$ instead of $M_0(t)$ (and doing a little extra work), one obtains .....

The Interaction Morawetz Estimate (IME)

$$\int_0^T \int_{\mathbb{R}^3} |u(x,t)|^4 \, dx \, dt \ \leq$$

$$C \, \| \, u(\cdot,0) \, \|^2_{L^2(\mathbb{R}^3)} \cdot \sup_{0 \leq t \leq T} \| \, (-\Delta_x)^{1/4} u(\cdot,t) \, \|^2_{L^2(\mathbb{R}^3)}$$

Rules Out	$\lvert u(x,t) \rvert \sim 1$
for	$\lvert x - x_0(t) \rvert \lesssim 1$.

<u>IME</u> (for $p = 3$):

$$\int_0^T \int_{\mathbb{R}^3} |u(x,t)|^4 \, dx, dt \;\leq\; C \parallel u(\cdot, 0) \parallel_{L^2}^2 \cdot \sup_{0 \leq t \leq T} \parallel (-\Delta_x)^{1/4} \, u(\cdot, t) \parallel_{L^2}^2$$

Note $(-\Delta_x)^{1/4} u$ takes only 1/2 an x-derivative.

Energy $= \frac{1}{2} \int |\nabla_x u(x,t)|^2 \, dx + \cdots$

takes a full x-derivative.

I-Team proved global existence for $p = 3$ NLS in Sobolev spaces in which the energy may be infinite.

Quintic Defocussing NLS ($p = 5$)

$$i\,\partial_t u + \Delta_x u = +|u|^4 u\,, \quad u|_{t=0} = u_0 \text{ given}$$

Critical for the Energy:

- Short-Time Solutions for finite energy.
- Global Solutions for small energy.

$$\text{Energy} = \tfrac{1}{2}\int_{\mathbb{R}^3} |\nabla_x u(x,t)|^2\,dx + \frac{1}{6}\int_{\mathbb{R}^3} |u(x,t)|^6\,dx.$$

The Challenge:

Prove Global Existence

for

Large Finite Energy

- Done by J. Bourgain for the radial case,

$$u(x,t) = f(x,t).$$

BIG, HARD, FORBIDDING!

- The general case is MUCH HARDER than the radial case.

Radial $\Longrightarrow$ Singularities of $u(x,t)$ can form only at $x = 0$.

General $\Longrightarrow$ Singularities may form anywhere.

The I-Team solved the general case, by proving ...

Theorem: For finite-energy initial data u_0, the Defocussing Quintic NLS has a global solution. If the initial data u_0 belong to $H^s(\mathbb{R}^3)$, $(s > 1)$, then so does the solution $u(\cdot, t)$ at any time t.

Moreover, there exist solutions u_+, u_- of the free Schrödinger equation $(i\partial_t + \Delta_x)u_\pm = 0$, such that

$$\int_{\mathbb{R}^3} |\nabla_x(u(x,t) - u_\pm(x,t))|^2 \, dx \longrightarrow 0 \text{ as } t \longrightarrow \pm\infty\,.$$

Ideas of Proof

- INTERACTION MORAWETZ ESTIMATE

  (with cutoffs, that make life much harder)

- Many ideas from Bourgain

  (especially INDUCTION on the ENERGY)

- Many ideas I can't begin to describe here.

Arithmetic of Progressions of Primes

(B. Green and T. Tao)

Theorem (B. Green - T. Tao):

Given $k \geq 3$, there exist k primes in arithmetic progression.

More precisely, given $k \geq 3$, there exist $c(k)$, $N_0(k) > 0$ s.t. for any $N \geq N_0(k)$ we have

$\#\{k$-term arithmetic progressions among the primes $\leq N\}$ is at least

$$\frac{c(k)\, N^2}{(\log N)^k}.$$

Szemerédi's Theorem (Sets):

Given k, δ, there exist $c(k, \delta)$, $N_0(k, \delta)$ s.t. for any $N \geq N_0(k, \delta)$, the following holds: Let $E \subset \mathbb{Z}_N$, with $\frac{\#E}{N} > \delta$.

Then E contains a k-term arithmetic progression. In fact,

$\#\{k$-term arithmetic progressions in $E\}$

$$\geq c(k, \delta)\, N^2 .$$

Szemerédi's Theorem (Functions):

Given k, δ, there exist $c(k, \delta)$, $N_0(k, \delta)$ s.t. for any $N \geq N_0(k, \delta)$, the following holds:

Let $f : \mathbb{Z}_N \longrightarrow \mathbb{R}$, with

- $0 \leq f(x) \leq 1$ (all $x \in \mathbb{Z}_N$), and
- $Av_{x \in \mathbb{Z}_N} f(x) \geq \delta$.

Then

$$Av_x, r \in \mathbb{Z}_N \{f(x) \cdot f(x+r) \ \ldots \ f(x+(k-1)r)\} \geq c(k, \delta)$$

Three Completely Different Proofs of Szemerédi's Theorem

- Szemerédi (Combinatorics)

- Furstenberg (Ergodic Theory)

- Gowers (Non-linear Fourier Analysis)

The proof of Green and Tao synthesizes all the previous work on Szemerédi's Theorem.

More precisely, it quotes Szemerédi's Theorem, and it uses ideas from the proofs of Furstenberg and Gowers.

IDEA: Try applying Szemerédi, with $\delta = 1/2$, to the function

$$f(x) = \left\{ \begin{array}{ll} \frac{1}{2} \log x & \text{if } x \text{ is a prime } < N \\ 0 & \text{otherwise} \end{array} \right\} \text{ on } \mathbb{Z}_N .$$

Note $Av_{x \in \mathbb{Z}_N} f(x) \sim \frac{1}{2}$ for large N;

Conclusion of Szemerédi says

$Av_{x,r} \{f(x)\, f(x+r) \cdots f(x+(k-1)r)\} \geq c(k, \frac{1}{2})$, i.e.,

$\#\{k\text{-term arithmetic prog. among primes} \leq N\} \geq \frac{c(k)N^2}{(\log N)^k}$

(exactly the desired result).

Oops!

$$f(x) = \left\{ \begin{array}{ll} \frac{1}{2}\log x & \text{if } x \text{ is a prime} \le N \\ \\ 0 & \text{otherwise} \end{array} \right\}$$

fails to satisfy the hypothesis

$$0 \le f(x) \le 1 \quad (\text{all } x \in \mathbb{Z}_N)$$

To get around this, ...

Green & Tao prove an extension of Szemerédi's Theorem, in which the hypothesis

$$0 \leq f(x) \leq 1$$

is replaced by

$$0 \leq f(x) \leq \nu(x)$$

for a suitable function $\nu(x)$, called a "pseudo-random measure" by Green & Tao.

Defining Conditions for a Pseudo-Random Measure

- $Av_{x \in \mathbb{Z}_N} \nu(x) = 1$

- Upper Bounds on certain averages

$$Av_{\vec{x}=(x_1,\cdots,x_t)\in(\mathbb{Z}_N)^t} \left\{ \prod_{i=1}^{m} \nu(\lambda_i(\vec{x})) \right\}$$

for certain affine functions

$$\lambda_1, \cdots, \lambda_m : (\mathbb{Z}_N)^t \longrightarrow \mathbb{Z}_N .$$

Green-Tao Version of Szemerédi's Theorem:

Given k, δ there exist $c(k, \delta)$, $N_0(k, \delta)$

s.t. for all $N > N_0(k, \delta)$ the following holds.

Let ν be a pseudo-random measure on $\mathbb{Z}_N$.

Let $f : \mathbb{Z}_N \longrightarrow \mathbb{R}$, with

- $0 \leq f(x) \leq \nu(x)$ (all $x \in \mathbb{Z}_N$)
- $Av_{x \in \mathbb{Z}_N} f(x) \geq \delta$.

Then

$$Av_{x,r \in \mathbb{Z}_N} \{f(x) f(x+r) \cdots f(x+(k-1)r)\} \geq c(k, \delta).$$

First Discuss Proof of

“Green-Tao-Szemerédi Theorem”,

Then Come Back & Apply it to Primes

Proof of Green-Tao-Szemerédi

- Split $f = f_U + f_{U^\perp}$ ("uniform" + "anti-uniform")

- Write $Av_{x,r\in\mathbb{Z}_N}\{f(x)\, f(x+r)\, \cdots\, f(x+(k-1)r)\}$ as a sum of terms

(*) $Av\{f_0(x)\, f_1(x+r)\, \cdots\, f_{k-1}(x+(k-1)r)\}$,

with each $f_i = f_U$ or $f_{U^\perp}$.

- Any term (*) in which at least one $f_i = f_U$ is negligibly small, thanks to ideas that go back to Gowers' proof of Szemerédi's Theorem.

- This leaves us with a single "MAIN TERM", namely

$$Av_{x,r\in\mathbb{Z}_N}\left\{f_{U^\perp}(x)f_{U^\perp}(x+r)\cdots f_{U^\perp}(x+(k-1)r)\right\}$$

To handle the main term, Green and Tao partition $\mathbb{Z}_N$ into subsets $E_1, E_2, \cdots, E_A$, and replace $f_{U^\perp}$ by its average over each of the E_α.

Call this averaged function $\bar{f}_{U^\perp}$.

- Replacing $f_{U^\perp}$ by $\bar{f}_{U^\perp}$ in the MAIN TERM makes a negligibly small difference.

- $\bar{f}_{U^\perp}$ satisfies the hypotheses of the classic Szemerédi Theorem.

The above results on $\bar{f}_{U^\perp}$ follow from ideas that go back to Furstenberg's proof of Szemerédi's Theorem.

Now the Green-Tao-Szemerédi Theorem follows at once, by simply applying the classic Szemerédi Theorem to $\bar{f}_{U^\perp}$.

How Green-Tao-Szemerédi applies to the primes.

"Just" need to find $\nu(x)$ s.t.

- $f(x) \leq \nu(x)$, where

$$f(x) = \left\{ \begin{array}{ll} \frac{1}{2}\log x & \text{if } x < N \text{ is prime} \\ 0 & \text{otherwise} \end{array} \right\}$$

and

- $\nu(x)$ is a pseudo-random measure.

Such a ν comes from work of Goldston-Yildirim using not-so-hard analytic number theory.

Much More First-Rate Work, including ...

- Program to study wave maps via harmonic maps
- Solution of Saturation Conjecture from representation theory.
  (Joint with Knutsen) ALGEBRA!

- WHAT NEXT?